Solution / marking scheme – Water and Objects (10 pt)

General rules
- In the following, "coefficients" refer to the numerical factors and do not include parameters.

Part A. Merger of water drops (2.0 pt)

A.1 (total 2.0 pt)

(2.0 pt)

$v = 0.23$ m/s
- No deduction if the answer falls within the range 0.22 m/s $\le v \le 0.24$ m/s

partial points

The surface energy per drop before the merger:

$$(0.4 \text{ pt}) \quad E = 4\pi a^2 \gamma \tag{A.1.1}$$

The surface energy difference:

$$(0.6 \text{ pt}) \quad \Delta E = 4\pi \left(2 - 2^{2/3}\right) a^2 \gamma \tag{A.1.2}$$

The transfer of surface energy to kinetic energy :

$$(0.4 \text{ pt}) \quad Mv^2/2 = k\Delta E \tag{A.1.3}$$

where $M = 4\pi a^3 \rho/3 \times 2 = 8\pi a^3 \rho/3$ is the mass of the drop after the merger.
- No partial point will be given if the factor k is missing.

Numerical evaluation:

$$v = \sqrt{\frac{2k\Delta E}{M}} = \sqrt{3\left(2 - 2^{2/3}\right)\frac{k\gamma}{\rho a}} = \sqrt{3\left(2 - 2^{2/3}\right) \times \frac{0.06 \times \left(7.27 \times 10^{-2}\right)}{\left(1.0 \times 10^{3}\right) \times \left(100 \times 10^{-6}\right)}} = 0.23\cancel{2} \text{ m/s}$$

Part B. A vertically placed board (4.5 pt)

B.1 (total 0.6 pt)
Usable letters: ρ, g, z, P_0

(0.6 pt)

$$P = P_0 - \rho g z$$

- No point will be given for $P = P_0 + \rho g z$

Commentary

The expression, $P = P_0 - \rho g z$, holds for both $z < 0$ and $z > 0$, as long as z is inside the water.

B.2 (total 0.8 pt)
Usable letters: ρ, g, z_1, z_2

(0.8 pt)

$$f_x = \frac{1}{2}\rho g\left(z_2^2 - z_1^2\right)$$

- Give 0.6 pt for $f_x = \rho g\left(z_2^2 - z_1^2\right)$
- Give 0.4 pt for $f_x = \frac{1}{2}\rho g\left(z_1^2 - z_2^2\right)$

Commentary

Because the atmospheric pressure P_0 exerts no net horizontal force on the water block, we have

$$f_x = \int_{z_2}^{z_1}(-\rho g z)dz = \frac{1}{2}\rho g\left(z_2^2 - z_1^2\right)$$

B.3 (total 0.8 pt)
Usable letters: $\gamma, \theta_1, \theta_2$

(0.8 pt)

$$f_x = \gamma\cos\theta_1 - \gamma\cos\theta_2$$

- Give 0.6 pt for $f_x = \gamma\cos\theta_2 - \gamma\cos\theta_1$
- Give 0.4 pt for $f_x = \gamma\cos\theta_2 + \gamma\cos\theta_1$ or $f_x = -\gamma\cos\theta_2 - \gamma\cos\theta_1$.

B.4 (total 0.8 pt)

(0.4 pt)

$$a = 2$$

- No point will be given for $a \neq 2$.

Usable letters: γ, ρ

(0.4 pt)

$$\ell = \sqrt{\frac{\gamma}{\rho g}}$$

- If an unnecessary coefficient is included as a factor, 0.2 pt will be deducted.

B.5 (total 1.5 pt)
Usable letters: $\tan\theta_0, \ell$

(1.5 pt)

$$z(x) = -\ell \tan\theta_0 e^{-x/\ell}$$

- Deduct 0.2 pt for $z(x) = -\ell \sin\theta_0 e^{-x/\ell}$ or $z(x) = -\ell\theta_0 e^{-x/\ell}$.

partial points

$z' = \tan\theta$ leads to

(0.2 pt) $$\cos\theta = \frac{1}{\sqrt{1+(z')^2}} \tag{B.5.1}$$

(0.1 pt) $$\cos\theta \simeq 1 - \frac{1}{2}(z')^2 \tag{B.5.2}$$

Plug this into Eq.(1) to obtain,

(0.2 pt) $$\frac{z^2}{\ell^2} - z'^2 = \text{const.} \tag{B.5.3}$$

Take the derivative of both sides with respect to x :

(0.5 pt) $$z'' = \frac{z}{\ell^2} \tag{B.5.4}$$

which is the differential equation which determines the water surface form.
General solution:

(0.2 pt) $$z = Ae^{x/\ell} + Be^{-x/\ell} \tag{B.5.5}$$

The boundary condition, $z(\infty) = 0$, leads to

(0.1 pt) $$A = 0 \tag{B.5.6}$$

The boundary condition, $z'(0) = \tan\theta_0$, leads to

(0.2 pt) $$B = -\ell\tan\theta_0 \tag{B.5.7}$$

Part C. Interaction between two rods (3.5 pt)

C.1 (total 1.0 pt)
Usable letters: $\theta_{\mathrm{a}}, \theta_{\mathrm{b}}, z_{\mathrm{a}}, z_{\mathrm{b}}, \rho, g, \gamma$

(1.0 pt)

$$F_x = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right) + \gamma\left(\cos\theta_{\mathrm{b}} - \cos\theta_{\mathrm{a}}\right)$$

- Give 0.8 pt for $F_x = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right) + \gamma\left(\cos\theta_{\mathrm{a}} - \cos\theta_{\mathrm{b}}\right)$
- Give 0.6 pt for $F_x = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right) + \gamma\cos\theta_2 + \gamma\cos\theta_1$ or $F_x = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right) - \gamma\cos\theta_2 - \gamma\cos\theta_1$.

partial points

The holizontal component of the force due to the pressure is

$$\text{(0.6 pt)} \quad \int_{z_{\mathrm{a}}}^{z_{\mathrm{b}}} (\rho g z)\, dz = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right) \qquad \text{(C.1.1)}$$

Commentary

Comment 1: How to apply the experience in B.1 is as follows. Let z_{bottom} the z-coordinate at the bottom of the rod, then from the discussion in B1, we see

$$F_x = \int_{z_{\mathrm{bottom}}}^{z_{\mathrm{a}}} (-\rho g z) dz + \left(-\int_{z_{\mathrm{bottom}}}^{z_{\mathrm{b}}} (-\rho g z) dz\right) = \int_{z_{\mathrm{a}}}^{z_{\mathrm{b}}} (\rho g z) dz$$

Comment 2: The fact that the contribution due to the pressure does not depend on the shape of the cross-section can be demonstrated as follows. The pressure at the point s on the contour C along the cross-sectional boundary is

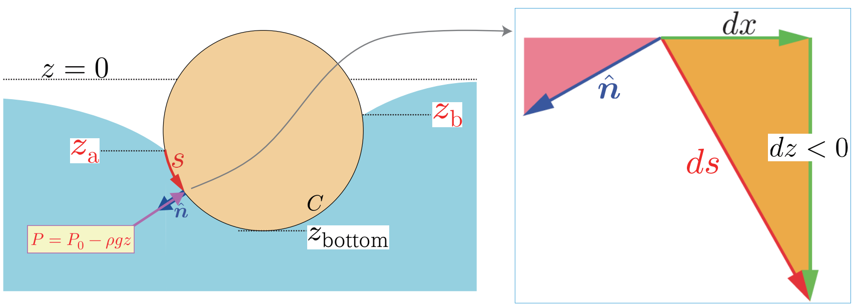
$$-P\hat{n}ds = (-P_0 + \rho g)\hat{n}ds.$$

Let $\hat{x}$ the unit vector pointing the positive x-direction and noting $\hat{x}\cdot\hat{n}ds = dz$ (see the figure shown below), the holizontal component becoms and its holizontal component becomes

$$-P\hat{n}\cdot\hat{x}ds = -P_0 dz + \rho g dz.$$

Integrating along the contour C, we obtain

$$\oint_C (-P\hat{n}\cdot\hat{x}ds) = \int_{z_{\mathrm{a}}}^{z_{\mathrm{b}}} (\rho g z) dz = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right)$$



C.2 (total 1.5 pt)
Unusable letters: $\theta_\mathrm{a}, \theta_\mathrm{b}, z_\mathrm{a}, z_\mathrm{b}$

(1.5 pt)

$$F_x = -\frac{1}{2}\rho g z_0^2$$

- Give 1.3 pt for $F_x = -\rho g z_0^2$.
- Give 0.8 pt for $F_x = \frac{1}{2}\rho g z_0^2$.

partial points

Apply the boundary conditions to Eq. (1) to obtain

$$\text{(0.6 pt)} \quad \underbrace{\frac{1}{2}\rho g z_\mathrm{a}^2 + \gamma\cos\theta_\mathrm{a}}_{x=x_\mathrm{a}} = \underbrace{\frac{1}{2}\rho g z_0^2 + \gamma}_{x=0} \tag{C.2.1}$$

- Give 0.4 pt for $\rho g z_\mathrm{a}^2 + \gamma\cos\theta_\mathrm{a} = \rho g z_0^2 + \gamma$

$$\text{(0.6 pt)} \quad \underbrace{\frac{1}{2}\rho g z_\mathrm{b}^2 + \gamma\cos\theta_\mathrm{b}}_{x=x_\mathrm{b}} = \underbrace{\gamma}_{x\to\infty} \tag{C.2.2}$$

- Give 0.4 pt for $\rho g z_\mathrm{b}^2 + \gamma\cos\theta_\mathrm{b} = \rho g z_0^2$

F_x is obtained by subtracting (C2.1) from (C2.2).

C.3 (total 1.0 pt)
Usable letters: $x_{\mathrm{a}}, z_{\mathrm{a}}$

(1.0 pt)

$$z_0 = \frac{2z_{\mathrm{a}}}{e^{x_{\mathrm{a}}/\ell} + e^{-x_{\mathrm{a}}/\ell}}$$

- Correct alternative answer: $z_0 = \dfrac{z_{\mathrm{a}}}{\cosh(x_{\mathrm{a}}/\ell)} = z_{\mathrm{a}}\mathrm{sech}(x_{\mathrm{a}}/\ell)$

partial points

General solution: $z(x) = Ae^{x/\ell} + Be^{-x/\ell}$
Taking into account the left-right symmetry, we obtain,

$$(0.3\text{ pt}) \quad A = B \tag{C.3.1}$$

Boundary condition, $z(0) = z_0$ leads to

$$(0.3\text{ pt}) \quad A + B = z_0 \tag{C.3.2}$$

Find the coefficients:

$$(0.2\text{ pt}) \quad A = z_0/2 \tag{C.3.3}$$

$$(0.2\text{ pt}) \quad B = z_0/2 \tag{C.3.4}$$