# Solution / marking scheme – Characterizing Soil Colloids (10 points)

General rules

- In the following, “coefficients” refer to the numerical factors and do not include parameters.

## Part A. Analysis of motions of colloidal particles (1.6 points)

**A.1** (total 0.8 pt)

( 0.4 pt)

$$v_0 = \frac{I_0}{M}$$

partial points

(0.2 pt) $Mv_0 = I_0$ (A.1.1)

( 0.4 pt)

$$\tau = \frac{M}{\gamma}$$

- 0.4pt if the answers are $v_0 = M/\gamma$ and $\tau = I_0/M$.

partial points

(0.2 pt) $M\dot{v} = -\gamma v(t)$ (A.1.2)

**A.2** (total 0.8 pt)

( 0.6 pt)

$$v(t) = \sum_i \frac{I_i}{M} e^{-(t-t_i)/\tau}$$

- 0.4pt if $\frac{I_i}{M} e^{-(t-t_i)/\tau}$ is written. The subscript can be any dummy variable used in the summation symbol.
- 0.2pt if sum is taken (if $\Sigma$ is written).
- the range of sum is not considered here (even if it is wrong).
- $\tau = M/\gamma$ can be substituted.

( 0.2 pt)

the inequality specifying the range of $t_i$ that needs to be considered:
$0 < t_i < t$

- $<$ can be $\leq$ (full mark is given).
- 0.2pt (full mark) is given to $t_i < t$ (without $0 <$)
- No point is given to $t_i > 0$ solely.

## Part B. Effective equation of motion (1.8 points)

**B.1** (total 1.0 pt)

( 0.5 pt) Usable letters: $C, \delta, t$

$$\langle \Delta x(t) \rangle = 0$$

( 0.5 pt) Usable letters: $C, \delta, t$

$$\langle \Delta x(t)^2 \rangle = C\delta t$$

partial points

(0.3 pt) $$\Delta x(t) = \sum_{n=1}^{N} v_n \delta \tag{B.1.1}$$

- 0.2pt if $\delta$ is missing.

(0.2 pt) $$\langle \Delta x(t)^2 \rangle = \sum_{n=1}^{N} C\delta^2 = NC\delta^2 = C\delta t \tag{B.1.2}$$

- 0.2pt only if $C\delta t$ is written. 0.1pt if only $\sum_{n=1}^{N} C\delta^2$ or $NC\delta^2$ is written.

**B.2** (total 0.8 pt)

( 0.4 pt)

$$\alpha = -1$$

( 0.4 pt)

$$\beta = 1$$

## Part C. Electrophoresis (2.7 points)

**C.1** (total 0.5 pt)

( 0.5 pt) Usable letters: $v, \delta, n(x_0), \frac{dn}{dx}(x_0)$

$$N_+(x_0) = \frac{1}{2}n(x_0)v - \frac{1}{4}\frac{dn}{dx}(x_0)v^2\delta$$

- 0.3pt if $\delta$ or $A$ or both are multiplied unnecessarily (subtraction of 0.2pt)
- 0.4pt if either coefficient (or both) is wrong (subtraction of 0.1pt)
- 0.4pt if the sign of the second term is wrong (subtraction of 0.1pt)
- If more than one of the above mistakes are made, points to subtract accumulate.

partial points

$$\text{(0.3 pt)} \quad N_+(x_0) = \int_{x_0 - v\delta}^{x_0} \frac{n(x)}{2\delta}dx \qquad \text{or} \qquad N_+(x_0) = \frac{v}{2}n(x_0 - v\delta/2) \tag{C.1.1}$$

- 0.2pt if $\delta$ or $A$ or both are multiplied unnecessarily (subtraction of 0.1pt)
- 0.2pt if any coefficient is wrong (subtraction of 0.1pt)
- 0.2pt if the integration range is $\int_{x_0}^{x_0+v\delta}$ (subtraction of 0.1pt)
- 0.2pt if $N_+(x_0) = \frac{v}{2}n(x_0 + v\delta/2)$ (subtraction of 0.1pt)
- If more than one of the above mistakes are made, points to subtract accumulate.

**C.2** (total 0.7 pt)

( 0.4 pt) Usable letters: $C, \delta, n(x_0), \frac{dn}{dx}(x_0)$

$$J_D(x) = -\frac{1}{2}\frac{dn}{dx}(x)C\delta$$

- 0.3pt if the sign or the coefficient is wrong (but pay attention to carryover from C.1).

partial points

$$\text{(0.1 pt)} \quad N_-(x_0) = \frac{1}{2}n(x_0)v + \frac{1}{4}\frac{dn}{dx}(x_0)v^2\delta \tag{C.2.1}$$

( 0.1 pt) Usable letters: $C, \delta$

$$D = \frac{1}{2}C\delta$$

( 0.2 pt) Usable letters: $D, t$

$$\langle \Delta x(t)^2 \rangle = 2Dt$$

- No point if the answer includes $C$ or $\delta$.

**C.3** (total 0.5 pt)

( 0.5 pt) Usable letters: $n(x), T, Q, E, k$

$$\frac{dn}{dx} = \frac{n(x)}{kT} QE$$

partial points

(0.3 pt) $\Pi(x)A + n(x)A\Delta x QE = \Pi(x + \Delta x)A$ (C.3.1)

**C.4** (total 0.5 pt)

( 0.3 pt)

$$\langle v(t) \rangle = \frac{QE}{\gamma}(1 - e^{-t/\tau})$$

- $\tau = M/\gamma$ can be substituted.

partial points

(0.3 pt) $M\frac{d\langle v(t)\rangle}{dt} = -\gamma\langle v(t)\rangle + QE$ (C.4.1)

( 0.2 pt)

$$u = \frac{QE}{\gamma}$$

**C.5** (total 0.5 pt)

( 0.5 pt) Usable letters: $k, \gamma, T$

$$D = \frac{kT}{\gamma}$$

partial points

(0.2 pt) $J_D(x) = -\frac{DQE}{kT}n(x)$ (C.5.1)

(0.2 pt) $J_Q(x) = \frac{QE}{\gamma}n(x)$ (C.5.2)

## Part D. Mean square displacement (2.4 points)

**D.1** (total 1.0 pt)

( 1.0 pt)

$N_A = 5.6 \times 10^{23}\ \mathrm{mol}^{-1}$

- No reduction if the unit is missing.
- 0.8pt if the second digit is wrong but the value is in the range $5.5$–$5.7 \times 10^{23}$.

partial points

$$(0.5\text{ pt})\quad \langle \Delta x^2 \rangle = \frac{RT\Delta t}{3\pi a \eta N_A} \tag{D.1.1}$$

- 0.3pt if both the answer of C.2 ($\langle \Delta x^2 \rangle = 2D\Delta t$) and that of C.5 ($D = \frac{kT}{\gamma}$) are given in the worksheet for D.1. The combination of them ($\langle \Delta x^2 \rangle = \frac{2kT\Delta t}{\gamma}$) is also acceptable. $k = R/N_A$ and $\gamma = 6\pi a \eta$ can be substituted here.
- No reduction if $t$ is used for $\Delta t$.

$$(0.3\text{ pt})\quad \langle \Delta x^2 \rangle = 6.34\ \mu\mathrm{m}^2 \tag{D.1.2}$$

- No reduction if the value is in the range $6.2$–$6.4\ \mu\mathrm{m}^2$.
- 0.2pt if the value is in the range $4$–$9\ \mu\mathrm{m}^2$ or if the standard deviation of $\Delta x$ is in the range $2$–$3\ \mu\mathrm{m}$.
- Subtract 0.1pt if the unit is missing or wrong.

**D.2** (total 0.8 pt)

( 0.2 pt) Usable letters: $u, D, t$

$$\langle \Delta x^2 \rangle = (ut)^2 + 2Dt$$

( 0.2 pt)

$$\langle \Delta x^2 \rangle \propto \begin{cases} t & \text{for small } t \\ t^2 & \text{for large } t \end{cases}$$

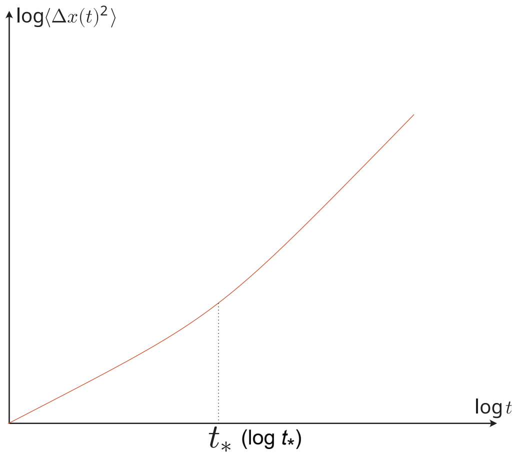
- 0.1pt independently for each answer.

( 0.2 pt)

$$t_* = \frac{2D}{u^2}$$

( 0.2 pt)

Points are given according to the criteria given below.




- 0.1pt if the graph is monotonically increasing and convex (no points if there are multiple curves that look like the answered graph)
- 0.1pt if $t_*$ is written between the two power-law regions (the label can be either $t_*$ or $\log t_*$).

**D.3** (total 0.6 pt)

( 0.6 pt)

$$\langle \Delta x^2 \rangle = \begin{cases} 2Dt & \text{for small } t \\ u_0^2 t^2 & \text{for intermediate } t \\ (u_0^2 \delta) t & \text{for large } t \end{cases}$$

- 0.2pt independently for each answer.
- Wrong answer in B.1 is not considered.

## Part E. Water purification (1.5 points)

**E.1** (total 1.5 pt)

( 1.5 pt)

$$c = \frac{8B^2\epsilon^3(kT)^5}{e^4N_AA^2q^6}$$

- 1.3pt if only the coefficient is wrong ($e$ is a part of the coefficient) (then no further partial point is given)

partial points

$$\text{(0.5 pt)} \quad \min U'(d) = 0 \tag{E.1.1}$$

- No point for $U'(d) = 0$ solely (without indicating what $d$ to consider) or $U'(a) = 0$.
- 0.2pt if the graph of the potential with an energy barrier (the graph first increases monotonically, then decreases monotonically) is drawn (this is the potential for $c < c_*$)
- independently, 0.2pt if the graph of the potential without an energy barrier (the graph increases monotonically) is drawn (this is the potential for $c > c_*$)

$$\text{(0.2 pt)} \quad U'(d) = \frac{A}{d^2} - \frac{B\epsilon(kT)^2}{q^2\lambda}e^{-d/\lambda} = 0 \tag{E.1.2}$$

$$\text{(0.2 pt)} \quad U''(d) = -\frac{2A}{d^3} + \frac{B\epsilon(kT)^2}{q^2\lambda^2}e^{-d/\lambda} = 0 \tag{E.1.3}$$

- 0.2pt (out of the 0.4pt right above) if both $U'(d) = 0$ and $U''(d) = 0$ are written as simultaneous equations, without their correct explicit forms.

$$\text{(0.2 pt)} \quad d = 2\lambda = \sqrt{\frac{Aq^2\lambda}{B\epsilon(kT)^2}} \tag{E.1.4}$$

$$\text{(0.3 pt)} \quad \lambda = \frac{e^2Aq^2}{4B\epsilon(kT)^2} \tag{E.1.5}$$

- 1.4pt is given in total if (E.1.5) is written.
- 1.2pt if only the coefficient is wrong ($e$ is a part of the coefficient)

**E.1 (cont.)**

Another solution: it is also physically reasonable to consider $\max U(d) = 0$ instead of (E.1.1), though this does not meet the requirements given in the question. Therefore, partial points may be given as follows if the question is answered along this line.

partial points

$$(0.5\ \text{pt}) \quad \max U(d) = 0 \tag{E.1.6}$$

- No point for $U(d) = 0$ solely (without indicating what $d$ to consider) or $U(a) = 0$.
- 0.2pt if the graph of the potential with an energy barrier that is higher than $U = 0$ or $U(d \to \infty)$ is drawn (this is the potential for $c < c_*$)
- independently, 0.2pt if the graph of the potential with an energy barrier that is lower than $U = 0$ or $U(d \to \infty)$ is drawn (this is the potential for $c > c_*$)

$$U(d) = -\frac{A}{d} + \frac{B\epsilon(kT)^2}{q^2} e^{-d/\lambda} = 0 \tag{E.1.7}$$

$$(0.2\ \text{pt}) \quad U'(d) = \frac{A}{d^2} - \frac{B\epsilon(kT)^2}{q^2\lambda} e^{-d/\lambda} = 0 \tag{E.1.8}$$

- No point for (E.1.7)
- 0.2pt if both $U(d) = 0$ are $U'(d) = 0$ are written as simultaneous equations

$$(0.5\ \text{pt}) \quad d = \lambda = \frac{eAq^2}{B\epsilon(kT)^2} \tag{E.1.9}$$

- 1.2pt is given in total if (E.1.9) is written.
- 1.0pt if only the coefficient is wrong ($e$ is a part of the coefficient)

$$(0.1\ \text{pt}) \quad c = \frac{B^2\epsilon^3(kT)^5}{2e^2 N_A A^2 q^6} \tag{E.1.10}$$

- 1.3pt is given in total if (E.1.10) is written.
- 1.1pt if only the coefficient is wrong ($e$ is a part of the coefficient)