IPhO Lithuania 2021

Theoretical Examination

2. Electrostatic lens

10 points

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Goal

- A study of an idealized model of an electrostatic lens (inspired by recent developments in electron microscopes):
  - Finding the electrostatic potential by direct integration
  - Finding the electrostatic potential via the application of Gauss's law
  - Focusing of electrons by a charged ring
  - A thin-lens equation applied for the electrostatic lens
  - Finding the capacitance of a metallic ring
  - Determining the dynamics of charging/decharching

Part A (1 point)

Electrostatic potential on the axis of the ring

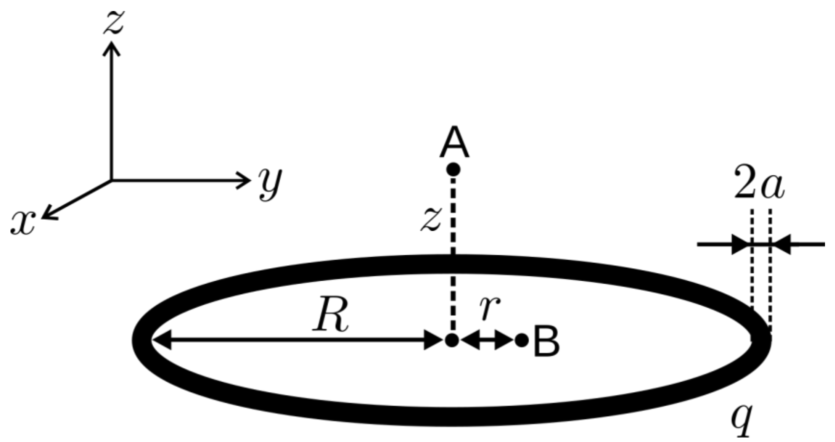
Consider a thin ring of radius R with a total charge q. The thickness of the ring $2a$ can be neglected.

A.1. find the electrostatic potential $\Phi(z)$ on the axis of the ring (0.3 pt.)

A.2. find the electrostatic potential to lowest non-zero order of r (0.4 pt.)

A.3. force acting on the electron for small z; what is the sign of q so that the motion is oscillatory? (0.2 pt.)

$$q > 0$$



A.4. angular frequency of harmonic oscillations (0.1 pt.)

simple “warm-up”

Part B (1.7 points)

Electrostatic potential in the plane of the ring

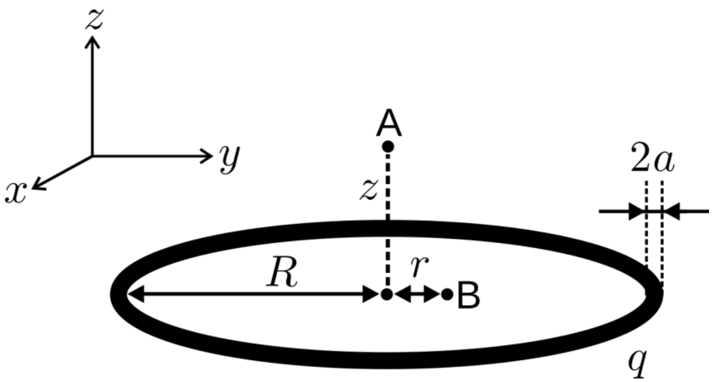
Consider the electrostatic potential in the plane of the ring. The potential is given by $\Phi(r) = q(\alpha + \beta r^2)$.

B.1. find the expression for β (1.5 pt.); integration or Gauss's law

- direct integration: Taylor expansion
- Gauss's law: students have to realize that the potential as a function of z and r is of the type $\Phi(z, r) = q(\alpha + \beta r^2 + \gamma z^2)$ for small z and r (no terms zr due to inversion symmetry)

B.2. force acting on the electron for small r; what is the sign of q so that the motion is oscillatory? (0.2 pt.)

$q < 0$

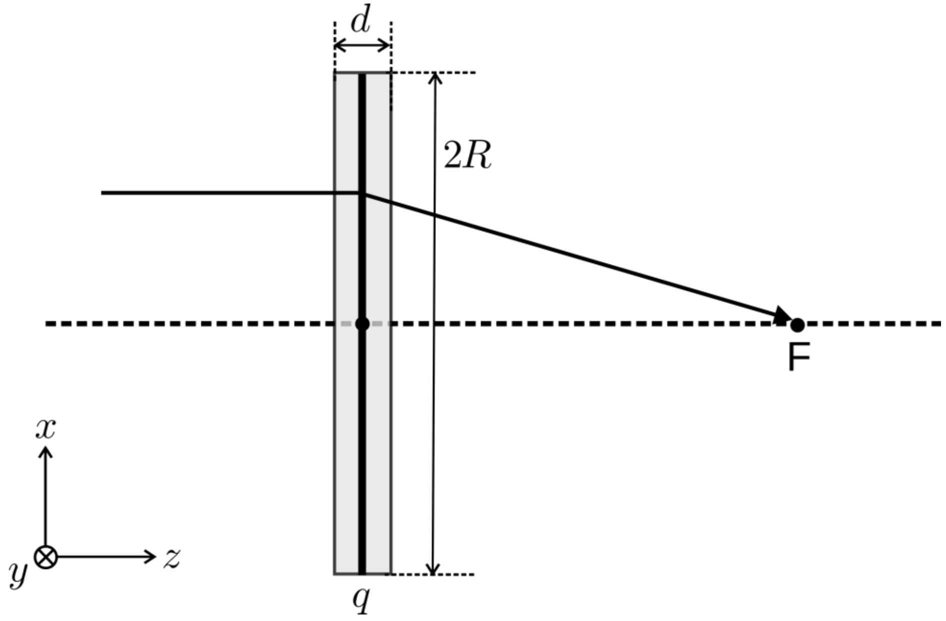


Part C (2.3 points)
Electrostatic lens with instantaneous charging

A model of an electrostatic lens: the ring is charged with q when electrons are in the "active region", otherwise the charge on the ring is 0. A source of electrons emits particles at precisely controlled moments.

C.1. Determine the focal length f of such a lens ($d \ll f$) (1.3 pt.)

Electrons are paraxial ($r \ll R$) and parallel to the z-axis.
Answer should be expressed in terms of the quantity β from B1 (the solution of B1 is not needed).



Part C (2.3 points)

Electrostatic lens with instantaneous charging

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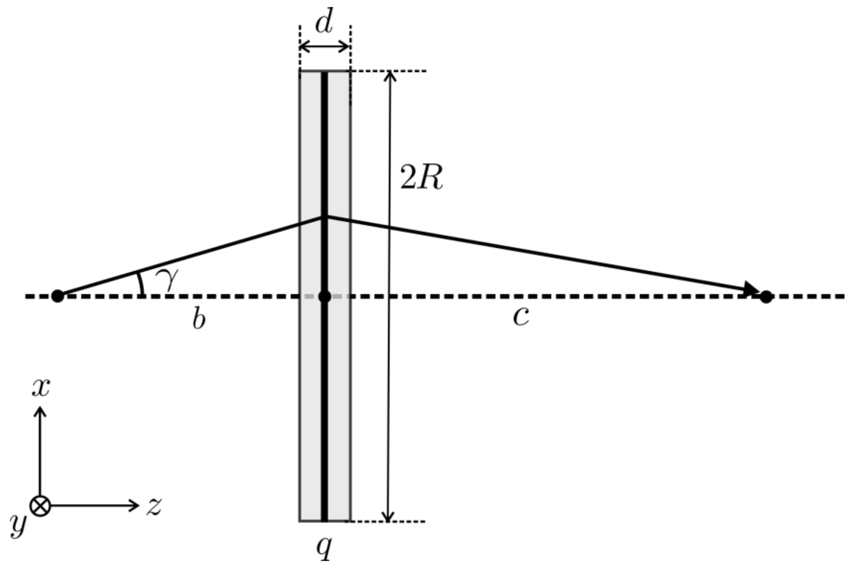
C.2. Electrons are no longer parallel to the z-axis, but are emitted at a small angle $\gamma \ll 1$ to the z-axis. The electron source is at a distance $b > f$ from the center of the ring; electrons are focused a distance c from the center of the ring.

Find c by solving kinematic equations (0.8 pt.)

C.3. Is the equation of a thin optical lens $\frac{1}{b} + \frac{1}{c} = \frac{1}{f}$ fulfilled for the electrostatic lens? Show it by explicitly calculating $\frac{1}{b} + \frac{1}{c}$ (0.2 pt.)

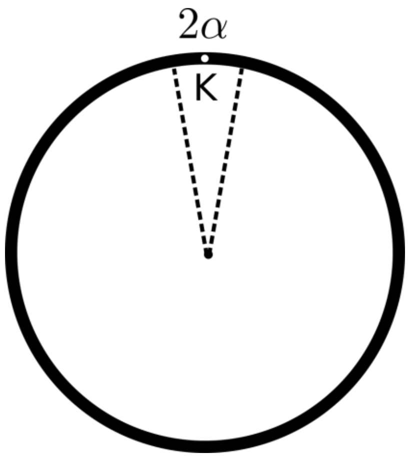


Part D (3 points)

The ring as a capacitor

D.1. Calculate the capacitance of the ring. Consider than the ring has a thickness $2a \ll R$ (2.0 pt.).

Suggestion: split the ring into two parts and calculate the potential in point K. The part with the angle 2α can be considered as a straight cylinder. In calculating the potential created by the rest of the ring one can ignore the thickness of the ring.



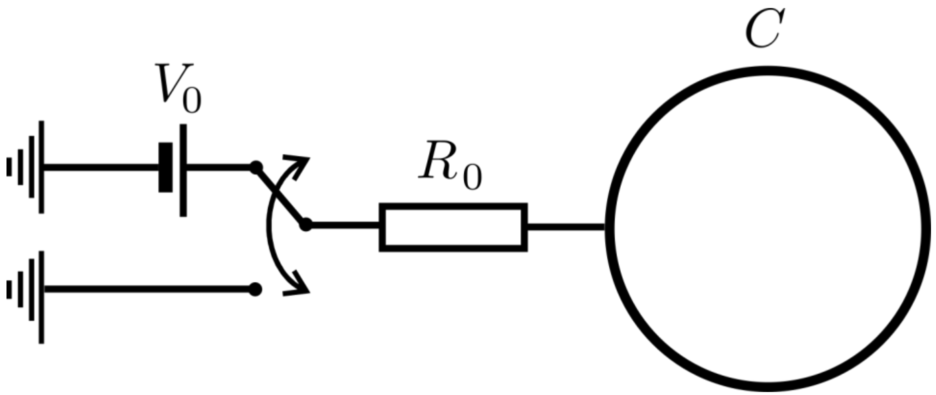
One will need these integrals that are given:

$$\int \frac{dx}{\sin x} = -\ln\left(\frac{\cos x + 1}{\sin x}\right) + \text{const} \qquad \int \frac{dx}{\sqrt{1+x^2}} = \ln\left(x + \sqrt{1+x^2}\right) + \text{const}$$

Part D (3 points)

The ring as a capacitor

D.2. When the electrons are in the “active region”, the ring is connected to a voltage source with a voltage V_0, otherwise the ring is connected to the ground. Time $t = 0$ corresponds to electrons in the plane of the ring. Determine the charge on the ring as a function of time. The sign of V_0 should be chosen so that the lens is focusing ($q < 0$). (1.0 pt.).



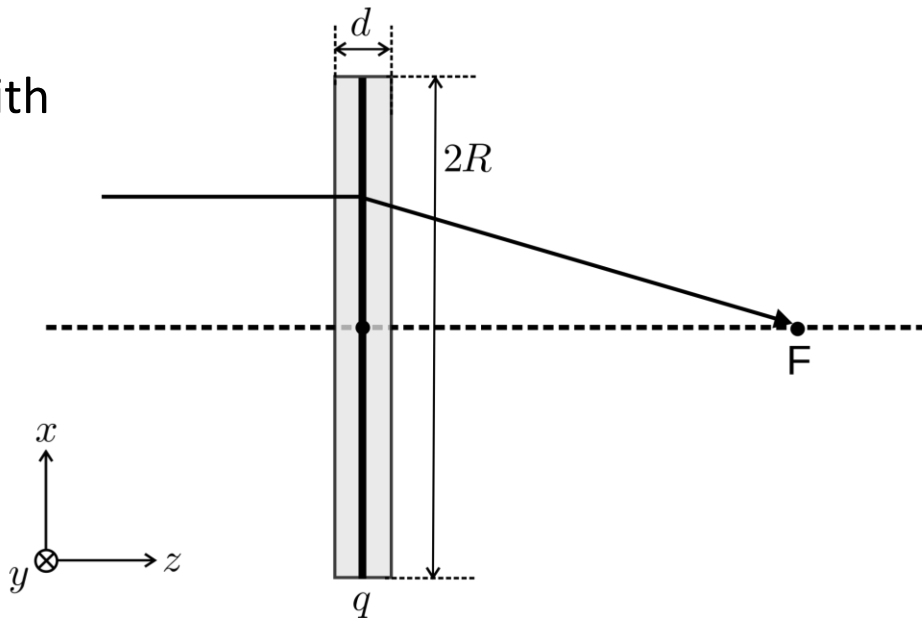
Part E (2 points)

Focal length of the lens with non-instantaneous charging

The charging of the lens is no longer instantaneous, but proceeds as determined in D.2.

E.1. Find the focal length of the lens f. Assume that $f/v \gg R_0C$, but d/v and R_0C are of the same order of magnitude (1.7 pt.). The thikness of the ring can be neglected.

E.2. The expression for f is similar to the one determined in part C, with q being substituted by a certain q_{eff}. Find the expression for q_{eff} (0.3 pt.)



Syllabus Coverage

Theoretical skills

2.1 General

- The ability to make appropriate approximations, while modelling real life problems (C1, C2, D1, E1)
- Recognition of and ability to exploit symmetry in problems (B1, Gauss’s law)

2.2 Mechanics

2.2.1 Kinematics

- Finding the focal length of lenses (C1, E1)
- Verifying the thin-lens expression by solving kinematic equations (C2)

Theoretical skills

2.3 Electromagnetic fields

2.3.1 Basic concepts

- Superposition principle for electrical fields (A1, A2, B1)
- Gauss’s law (B1)
- Kirchhhoff’s current law (D2)
- Boundary conditions at infinity (D1)

2.3.4 Circuits

- Linear resistors (D2)
- Time constants in RC circuits (D2, E2)

Theoretical skills

2.3 Oscillations and waves

- **2.4.1 Single oscillator**
  - Oscillations of an electron (A4, B2)
- **2.4.6 Geometrical optics and photometry**
  - Focal length of a lens (C1, E1)
  - Thin lens equations (C3)

Mathematics

- **4.1 Algebra**
  - Simplification of formulae by expansion (A2, B1)
- **4.2 Functions**
  - Simple equations involving trigonometric functions (D1)
- **4.7 Calculus**
  - Definite integrals of simple functions (B1, E1)
  - The application of definite integrals of more complicated functions (integrals themselves are given) (D1)
  - Calculating definite integrals by substitution (B1, D1, E1)
- **4.8 Approximate and numerical methods**
  - Taylor series (A2, B1)

Dependencies

$$
\begin{tikzcd}
 & & A1 \arrow[r] & A2 \arrow[r] & A3 \arrow[r] & A4 \\
 & & C1 \arrow[r] \arrow[dd, dashed] \arrow[ddr] & C2 \arrow[r] & C3 & \\
B1 \arrow[r] & B2 \arrow[ur, dashed] \arrow[dr, dashed] & & & & \\
 & & E1 \arrow[r] & E2 & & \\
 & & D1 & D2 \arrow[ul] & &
\end{tikzcd}
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A, B, C, and D essentially independent of each other (some knowledge gained in B2 is needed in C1 and E1 – the sign of q)