Planetary Physics (10 points)

General note: if student got the correct answer and the solution is physically and mathematically correct, all the points should be given.

Part A. Mid-ocean ridge (5.0 points)

A.1 (0.8 pt.)	Equating the two column pressures, $\rho g h = \rho_{\text{oil}} g h'$	0.2	
	Subtracting the two opposing forces due to different fluids, $F_x = F_1 - F_0$	0.1	
	Finding the two forces, $F_0 = \frac{\rho_0 g h^2 w}{2}$ and $F_1 = \frac{\rho_{\text{oil}} g h'^2 w}{2}$	0.3	
	if only one out of two formulae is correct		0.2
	Final expression, $F_x = \left(\frac{\rho_0}{\rho_{\text{oil}}} - 1\right) \frac{\rho_0 g h^2 w}{2}$	0.2	

A.2 (0.6 pt.) Direct calculation	Understanding that all three dimensions change according to the same law	0.1	
	New volume, $V = V_1 \left(1 - k_l \frac{T_1 - T}{T_1 - T_0}\right)^3$ or $V \approx V_1 \left(1 - 3k_l \frac{T_1 - T}{T_1 - T_0}\right)$	0.1	
	New density, $\rho(T) = \rho_1 \left(1 - k_l \frac{T_1 - T}{T_1 - T_0}\right)^{-3}$ or $\rho(T) \approx \rho_1 \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0}\right)$	0.1	
	Applying binomial/Taylor expansion at any stage of the solution	0.1	
	Final answer $\rho\,(T) \approx \rho_1 \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0}\right)$, $k = 3k_l$	0.2	
Using proportionality relations	$\rho \propto l^{-3}$	0.2	
	$l^{-3} \propto \left(1 - k_l \frac{T_1 - T}{T_1 - T_0}\right)^{-3} \approx \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0}\right)$	0.2	
	Final answer $\rho\,(T) \approx \rho_1 \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0}\right)$, $k = 3k_l$	0.2	

A.3 (1.1 pt.)	Equating the two column pressures, $p\,(0, h + D) - p\,(0, 0) = p\,(\infty, h + D) - p\,(\infty, 0)$	0.2	
	Column pressure at $x = 0$, $\rho_1 g\,(h + D)$	0.1	
	Column pressure far away, $p\,(\infty, h + D) - p\,(\infty, 0) = \rho_0 g h + \int_h^{h+D} \rho\,(T\,(\infty, z))\,g\,\mathrm{d}z$	0.2	
	Concluding that far away $\frac{\mathrm{d}T}{\mathrm{d}z} = \text{const.}$ if this is used but not explicitly stated – full marks	0.2	
	Obtaining $T\,(z) = T_0 + (T_1 - T_0)\,\frac{z-h}{D}$.	0.2	
	Final answer, $D = \dfrac{2}{k}\left(1 - \dfrac{\rho_0}{\rho_1}\right) h$	0.2	
A.4 (1.6 pt.)	Writing the force as a difference between force at the centre of the ridge and that at infinity marks allotted as long as subtraction of forces or pressures at correct x values is written	0.5	
	Correct pressure integration range $[z_1, z_2]$, where $z_1 \leq 0$ and $z_2 \geq h + D$	0.2	
	Pressure at the ridge axis, $p\,(0, z) = p\,(0, 0) + \rho_1 g z$	0.1	
	Pressure at infinity	0.4	
	formula in the ocean, $p\,(\infty, 0) + \rho_0 g z$		0.1
	formula in the crust, $p\,(\infty, 0) + \rho_0 g h + \int_{z'=h}^{z} \rho_1 \left(1 + k\frac{T_1 - T_0 - (T_1 - T_0)\frac{z'-h}{D}}{T_1 - T_0}\right) g\mathrm{d}z'$		0.3
	Final answer, $F \approx \frac{2gLh^2}{3k}\,(\rho_1 - \rho_0)\left(1 - \frac{\rho_0}{\rho_1}\right)$	0.4	
A.5 (0.9 pt.)	Stated variables on which the answer is going to depend: ρ_1, c, κ, D	0.2	
	Correct dimensions of c, $\mathsf{L}^2\mathsf{T}^{-2}\Theta^{-1}$	0.1	
	Correct dimensions of κ, $\mathsf{MLT}^{-3}\Theta^{-1}$	0.2	
	partial credit for writing down a correct law that can be used to find this, e.g. $\frac{\mathrm{d}P}{\mathrm{d}S} = \kappa\frac{\mathrm{d}T}{\mathrm{d}z}$, where P is the power conducted and S is the area of the surface across which the power is transferred		0.1
	Obtaining the simultaneous equations	0.2	
	Final answer $\tau \approx \frac{c\rho_1 D^2}{\kappa}$ or $\tau = A\frac{c\rho_1 D^2}{\kappa}$	0.2	

Part B. Seismic waves in a stratified medium (5.0 points)

B.1 (1.5 pt.)			
Method 1	Using the law of refraction in a stratified medium, $n(0)\sin\theta_0 = n(z)\sin\theta$	0.5	
	Definition of the refractive index, $n(z) \propto \frac{1}{v(z)}$	0.1	
	Identifying that at $\theta = \frac{\pi}{2}$, $z = R - R\sin\theta_0$	0.3	
	Finding $R = \frac{z_0}{\sin\theta_0}$	0.3	
	Final answer, $x_1(\theta_0) = 2z_0\cot\theta_0$ $x_1(\theta_0) = -2z_0\cot\theta_0$ is also correct if the alternative definition of θ_0 was used	0.3	
Method 2	Using the law of refraction in a stratified medium, $n(0)\sin\theta_0 = n(z)\sin\theta$	0.5	
	Definition of the refractive index, $n(z) \propto \frac{1}{v(z)}$	0.1	
	Obtaining the relation between dz and dθ, $\frac{\mathrm{d}z}{z_0}\sin\theta_0 = \cos\theta\,\mathrm{d}\theta$	0.3	
	Showing that ray path is an arc of a circle of radius $\frac{z_0}{\sin\theta_0}$	0.3	
	alternatively, obtaining the differential equation for $\frac{\mathrm{d}x}{\mathrm{d}\theta}$		0.3
	Final answer, $x_1(\theta_0) = 2z_0\cot\theta_0$ $x_1(\theta_0) = -2z_0\cot\theta_0$ is also correct if the alternative definition of θ_0 was used	0.3	

B.2 (1.5 pt.)	Realizing that $\frac{E}{\pi}$ is the energy emitted per unit angle	0.4	
	The idea to track an infinitesimal fraction of rays and relating an infinitesimal initial ray angle to the infinitesimal change in their destination, $\mathrm{d}\theta_0 = \frac{\mathrm{d}\theta_0}{\mathrm{d}x}\mathrm{d}x$	0.3	
	Applying the result of B.1 to obtain $\frac{\mathrm{d}x}{\mathrm{d}\theta_0} = -b\left(A + x^2\right)$	0.3	
	Final answer, $\varepsilon(x) = \frac{E}{\pi b(A^2+x^2)} = \frac{E}{\pi\left(4z_0^2+x^2\right)}$	0.1	
	Plotting $\varepsilon(x)$	0.4	
	maximum is at $x = 0$ and is of order $\frac{E}{A} \approx \frac{E}{z_0}$		0.1
	$\varepsilon(x) \longrightarrow 0$ as $x \longrightarrow \infty$		0.1
	$\frac{\mathrm{d}\varepsilon}{\mathrm{d}x} = 0$ at $x = 0$		0.1
	no stationary points at $x \neq 0$		0.1

B.3 (2.0 pt.)	The zone overlap condition, $N = \frac{x_1(\theta_0)}{x_- - x_+}$	1.0	
	considering the change in the zone width, $x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right) - x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right)$		0.4
	Writing $x_- - x_+$ in terms of $\frac{\mathrm{d}x_1}{\mathrm{d}\theta_0}$	0.4	
	Differentiating the result of B.1	0.2	
	$x_{\max} = N x_1(\theta_0)$	0.3	
	Final answer, $x_{\max} = \frac{Ab\cos^2(b\theta_0)}{\delta\theta_0} = \frac{2z_0\cos^2\theta_0}{\delta\theta_0}$.	0.1	